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LETTER TO THE EDITOR

A direct mapping between Eden growth model and directed polymers in random media

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Abstract. On the basis of preserving required symmetries and capturing the essential ingredients, the scaling properties of the frontier of Eden model clusters have been suggested by Kardar *et al* to be governed in the continuum limit by a Burgers equation with noise. We show that such a correspondence is exact by mapping explicitly the Eden model onto the problem of the conformation of a directed polymer in a random medium at zero temperature. The latter model has been shown to be itself in correspondence with Burgers' equation. In addition, this mapping allows us to have access to the distribution of noise to be included in Burgers' equation to exactly reproduce the Eden growth model. Finally, this implies that the Eden model always shows the same universal behaviour, in contrast to a recently found breakdown of universality for the directed polymer problem in a 'very disordered' medium.

It would be extremely presumptuous to try to find a field of physics where the heat equation has no relevance. A somewhat similar ubiquity seems to be encountered for the nonlinear diffusion equation of Burgers type:

$$\frac{\partial h(\boldsymbol{x}, t)}{\partial t} = A(\nabla h)^2 + B\Delta h + \eta \tag{1}$$

where $h(\boldsymbol{x}, t)$ is a function of the spatial coordinate $\boldsymbol{x}$ and time t, and $\eta(\boldsymbol{x}, t)$ is a noise term. In particular, Kardar *et al* [1] proposed to model the growth and roughening of the frontier of an Eden growth model [2] using this equation. h represents in this case the distance of the frontier from its origin after a time t. This equation—or a derived version of it—rules the conformation of directed polymers in a random medium at zero temperature as shown in [3]. The same problem in two dimensions can also be applied to the interface between two Ising domains at zero temperature when the coupling between sites fluctuates spatially (the so-called random bond problem) [3]. Other applications can be found in the geometrical structure of ballistic deposition fronts [4], or transport properties of random diode networks [5].

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One of the interesting features of (1) is that the function $h(x, t)$ exhibits non-trivial scaling behaviours. In particular the fluctuation w of h, $w^2 = \langle h(x, t)^2 \rangle - \langle h(x, t) \rangle^2$ (where $\langle \ldots \rangle$ denotes a *spatial* average), obeys the following scaling property:

$$w = L^{\chi} \varphi(L^z/t) \tag{2}$$

where L is the system size. The function φ satisfies the two limits: for $x \gg 1$, $\varphi(x) \sim x^{-\chi/z}$ or $w \propto t^{\chi/z}$ whereas for $x \ll 1$, $\varphi(x)$ is constant or $w \propto L^{\chi}$. The exponents χ and z only depend on the space dimension. We will refer in this paper to the case of a d-dimensional space plus one dimension of time as a $(d+1)$-dimensional problem. In $(1+1)$ dimension, the problem has been solved exactly, leading to the following values of $\chi = \frac{1}{2}$ and $z = \frac{3}{2}$, in the case of Gaussian noise η with short-range correlations in space and time. Medina *et al* [6] have discussed in detail the case of long-range space and time correlations.

Very recently, Zhang [7] has noticed that a problem very close to that of the directed polymer in a random medium can display different exponents χ and z if the noise distribution has a power-law tail at infinity. This numerical study was proposed to account for the discrepancy found between experimental determinations of χ and z on various experimental systems [8, 9] modelled as Eden clusters and the expected theoretical values mentioned above.

The purpose of this letter is first to establish a direct mapping between the Eden growth model and the directed polymer problem, and second to discuss the relevance of the breakdown of universality found by Zhang in the context of the Eden growth model [7].

Let us briefly recall the construction rules of Eden clusters. We consider a $(d+1)$-dimensional lattice, and choose a seed cluster. This cluster is often a single point, but any other set can be chosen. In order to avoid the problem of change of shape during the growth process, we will throughout this paper choose a d-dimensional hyperplane. For a finite-size system, we may implement periodic boundary conditions at the lateral borders of the initial seed, so as to deal with a 'cylindrical geometry'. We choose at random a site neighbour to the cluster, and include it in the cluster. We then update the set of sites which are susceptible to be chosen as the neighbours of the new cluster. We choose again a site in this set and include it in the cluster. This process is repeated *ad infinitum*. In order to define the time in this process we simply count the number of sites added to the cluster and divide by the area of the seed cluster. With this definition, during one time unit, on average, one site is added to the cluster per unit area. The clusters which are grown with this algorithm are compact. The average position of the frontier of the cluster is a hyperplane parallel to the seed, and moving away from the seed at a constant velocity of one (lattice mesh size per unit time).

The scaling properties of the set of frontier sites are accounted for by (2), where h represents the distance of the frontier site from the seed and w the fluctuations of h. Kardar *et al* [1] suggested modelling the growth of the Eden clusters by (1) in the continuum limit. The physical origin of the nonlinear contribution to (1) (which is the one that determines the scaling behaviour), is to be found in the assumption that the growth occurs normal to the *local* surface, whose orientation fluctuates.

Let us now define the directed polymer problem in a random medium, at zero temperature. We still consider a $(d+1)$-dimensional lattice, and assign to each site i a random number x_i picked from a distribution $f(x)\,dx$. These random numbers can be thought of as a local interaction energy between the polymer and the embedding medium at this site. A directed polymer on a lattice will be a directed path P along a

given direction (with no overhangs) that spans the lattice. The energy of the polymer having a conformation P will be the sum of x's along the path. At zero temperature, the polymer will be in the lowest energy conformation, and thus its energy will be E such that

$$E = \min_{P}\left\{\sum_{i\in P} x_i\right\}. \tag{3}$$

The path which satisfies this minimum is called P^*.

Different properties of the optimal conformation have been studied. In particular, the energy E can be shown to have a non-analytic correction term, as a function of the length of the polymer, l [1]

$$E = le_0 + \Delta E = l(e_0 + al^{-1/\nu_\parallel}) + \text{HOT} \tag{4}$$

where $\nu_\parallel$ is a critical exponent ($\nu_\parallel = \frac{3}{2}$ in two dimensions). The fluctuations of the energy around its mean value decrease with l as $l^{-1/\nu_\parallel}$. This scaling form has to be changed when the polymer is confined into a strip-like geometry, where the width of the network L is much smaller than the length l. In this case, the correction term scales as $L^{-1/\nu_\perp}$ instead of $l^{-1/\nu_\parallel}$. In two dimensions ($d = 1$) the value of $\nu_\perp$ is 1. The crossover from one correction term to the other may be expressed through the equation

$$\frac{\Delta E}{l} = L^{-1/\nu_\perp}\phi(L^{\nu_\parallel/\nu_\perp}/l). \tag{5}$$

The crossover function $\phi(x)$ in (5) behaves as $x^{1/\nu_\parallel}$ for large arguments and as a constant for small values of x. From this expression we may infer the 'roughness' of the minimum-energy path P^* (i.e. its mean deviation from a straight line δ) which increases with its length l according to a power law:

$$\delta \propto l^{\nu_\perp/\nu_\parallel} \tag{6}$$

as long as $\delta < L$. Thus the crossover between the two regimes reported above corresponds to the condition that the path P^* 'feels' or not the lateral boundaries (L equal to or larger than δ).

In the mapping of Kardar *et al* [3] between the directed polymer problem and Burgers' equation for h, it was the minimal energy that was mapped onto h, interpreted as the height of the interface in the Eden model. We will in the following show a different but direct mapping between the Eden model and the directed polymer problem. The scaling exponents used in (5) are therefore different from those used in [3]. We will give the correspondence later.

It should also be noted that since the path is asymptotically flat, (i.e. δ/l tends to zero as l increases), the directedness requirement is not necessary. Even if we do not impose that the path is directed, it will—at least on a large scale—be directed. It means in practice that eventually the mean energy of the polymer e_0 may vary, but not the scaling exponents $\nu_\parallel$ and $\nu_\perp$ under the removal of the directedness constraint.

Let us consider the Eden model, and construct an equivalent directed polymer problem. Since the clusters are compact, all sites of the lattice will after some time be part of the cluster. We set a 'clock' on each site and record the time during which a site has been a potential growth site, i.e. the time delay during which the site was neighbour to the cluster without being part of it. Let us call τ_i this delay for site i. It is easy to compute from these numbers the real time t_i at which the site became part of the cluster. It is equal to τ_i plus the time t_i' at which i became a neighbour to the

cluster. This latter time, t_i', is the minimum of t_j for all sites j neighbour to i. Writing this property recursively allows us to reformulate the time t_i as the minimum of the sum of delays τ_j along any path, P_i, connecting i to the initial cluster:

$$t_i = \min_{P_i}\left\{\sum_{j\in P_i}\tau_j\right\}. \tag{7}$$

We see a close resemblance of the latter equation to (3) relative to the directed polymer (DP) problem. The total time t_i is to be compared with the minimum energy of a polymer starting in i and reaching the seed cluster. We can construct a 'dictionary' which allows us to translate one property from one problem into the other one. Table 1 shows this translation both for the physical parameters and the corresponding critical exponents (as we will derive below). For instance, using the properties recalled above for the DP problem, we can state that, on average, the mean time needed to reach a site i, t_i, will be proportional to the distance h_i from i to the seed cluster as can be expected from the constant velocity of the interface. This corresponds to the leading term in the energy of the polymer as a function of its length. Thus naturally the τ distribution is normalized so as to ensure $e_0 = 1$ in (4).

Table 1. Translation between the physical parameters and the critical exponents of the Eden model and of the directed polymer (DP) problem.

Parameters		Critical exponents	
Eden	DP	Eden	DP
h	l	χ	$(\nu_\parallel - 1)/\nu_\perp$
L	L	z	$\nu_\parallel/\nu_\perp$
τ	x	$z/(z-\chi)$	$\nu_\parallel$
t	E	$1/(z-\chi)$	$\nu_\perp$

Let us now derive the relations between the scaling behaviours of both problems. The Eden cluster constructed up to a time t can be interpreted as the set of sites i such that the directed polymer starting in i and reaching the seed cluster has an energy less than or equal to t. The major difference between both problems is that, for the directed polymer problem, the control parameter is the length l, whereas it is the time t (corresponding to the energy for the polymer problem) for the Eden case. Let us first consider the early stage of the Eden model ($t \ll L^z$ in (2)). To map one problem onto the other we have to invert (4) so as to express l as a function of E:

$$l = E/e_0 - a/e_0(E/e_0)^{(\nu_\parallel - 1)/\nu_\parallel} + \text{HOT} \tag{8}$$

or using the dictionary, and setting $e_0 = 1$ as obtained above,

$$h = t - at^{(\nu_\parallel - 1)/\nu_\parallel} + \text{HOT}. \tag{9}$$

The fluctuation of h, or the roughness w of the interface, is given by the correction term of (9). Thus we may identify

$$\chi/z = (\nu_\parallel - 1)/\nu_\parallel. \tag{10}$$

To obtain a second relation between the critical exponents, we have to consider the cross-over criterion in both cases, or equivalently the late stage of the Eden cluster compared with a directed polymer feeling the lateral boundaries. For Eden, the

crossover size corresponds to $L \propto t^{1/z}$ (see (2)). For the directed polymer problem, (6) can be rewritten as $L \propto l^{\nu_\perp/\nu_\parallel}$. Identification of these two equations (using $l \propto t$ to the first order) leads to

$$z = \nu_\parallel / \nu_\perp . \tag{11}$$

Equations (10) and (11) makes it possible to express z and χ as a function of $\nu_\parallel$ and $\nu_\perp$, and *vice versa* as reported in table 1.

In this connection between both problems, we have also obtained the distribution of waiting times τ or equivalently of local interactions x. It has been shown [10] that the density n of growth sites per unit area in the Eden model is constant as soon as a very first transient regime is over. This early stage is very small compared with the crossover time which controls the roughness of the surface. The probability of picking a site during the time dt is simply dt/n, independent of the past history. The distribution of waiting time $f(\tau)\, d\tau$ is thus a Poisson distribution:

$$f(\tau) = (1/n)\, e^{-\tau/n} \tag{12}$$

for $0 \leqslant \tau < \infty$. Thus the distribution of local energy x in the directed polymer problem which reproduces the Eden growth model is completely characterized from our construction. Figure 1 shows the result of a small numerical simulation of the direct measurement of the waiting time distribution during the growth of an Eden cluster. The semi-log plot of the distribution gives a straight line which agrees with the result (12).

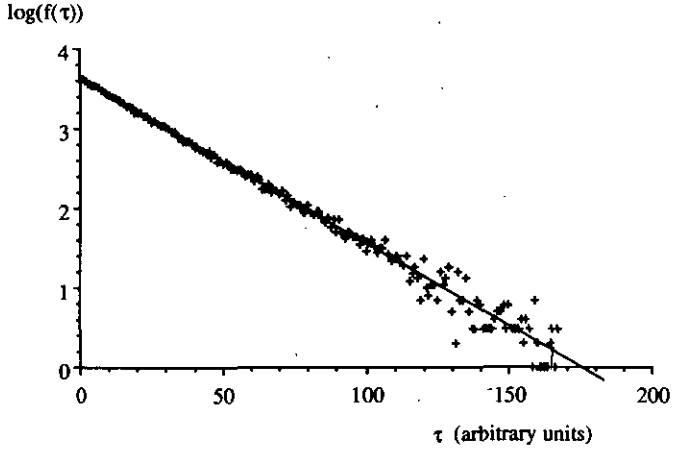

Figure 1. Semi-log plot of the distribution of waiting times τ from the Eden model. The simulation points (+) are compared with the expected distribution (straight line).

Let us finally discuss the problem of universality. Zhang [7] recently considered a problem extremely close to the directed polymer model as described above. The only difference lies in the fact that the optimal path studied was the one which *maximized* the sum of random variables x of the sites through which the path goes. Obviously when the distribution of x is bounded, both the minimal and the maximal path have similar properties since one can change x into $-x$ (plus eventually a constant to keep the local energies positive). However, the observation of Zhang was to note that when the x distribution presents a power-law tail at infinity with a small exponent α, then

the scaling exponents $\nu_\perp$ and $\nu_\parallel$ changed continuously with α. This property was called 'breakdown of universality'. The motivation to consider this breakdown was to propose an explanation for the discrepancy between the observed values of χ and z in experiments [8, 9] which were apparently related to the Eden model, and the 'classical values' of χ and z.

In the mapping we have presented, we note that:

- it is the *minimal* path that comes naturally into play in the polymer problem,
- the distribution of local energy x or τ is uniquely determined as an exponential distribution, and does not present any power-law tail close to infinity.

The first remark is the more important. It has been shown [11] on a hierarchical lattice (which mimics a two-dimensional Euclidean one) that all possible stable scaling regimes could be worked out. In particular, when only positive local energies are considered ($x > 0$)—what is obviously expected if x is interpreted as a delay time—then the minimal path problem has a unique basin of attraction, i.e. all distributions of x lead to the same values of χ and z. No breakdown of universality is ever seen.

This does not exclude the fact that for other problems which could be modelled by a minimal path in the directed polymer model, a breakdown of universality occurs provided the distribution of local energies has a power-law tail close to minus infinity with an exponent close enough to zero (less than six [11]) as observed by Zhang [7].

We have shown that the Eden model could be mapped exactly onto a directed polymer problem, including the distribution of local energies in the form of an exponential distribution. This allows us to establish that the Eden model has universal properties, and that no breakdown of universality is expected without changing the rules of the model.

This leaves open the question of the experimental observations of the scaling exponents χ and z on various real systems, which differ from the ones relative to the strict Eden model.

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References

[1] Kardar M, Parisi G and Zhang Y C 1986 *Phys. Rev. Lett.* **56** 889
[2] Eden M 1961 *Proc. 4th Berkeley Symp. on Mathematics, Statistics and Probability* vol IV, ed F Neyman (Berkeley, CA: University of California Press) p 233
[3] Huse D A and Henley C L 1985 *Phys. Rev. Lett.* **54** 2708
Kardar M 1985 *Phys. Rev. Lett.* **55** 2923
Huse D A, Henley C L and Fisher D S 1985 *Phys. Rev. Lett.* **55** 2924
[4] See e.g. Meakin P and Jullien R 1987 *J. Physique* **48** 1651
[5] Roux S and Herrmann H J 1987 *Europhys. Lett.* **4** 1227
Roux S, Hansen A and Guyon E 1987 *J. Physique* **48** 2125
[6] Medina E, Hwa T, Kardar M and Zhang Y C 1989 *Phys. Rev.* A **39** 3053
[7] Zhang Y C 1990 *J. Physique* **51** 2129
[8] Rubio M A, Edwards C A, Dougherty A and Gollub J P 1989 *Phys. Rev. Lett.* **63** 1685; 1990 *Phys. Rev. Lett.* **65** 1388
[9] Vicsek T, Cserzö M and Horváth V K 1990 *Physica* **167A** 315
[10] Zabolitzky J G and Stauffer D 1986 *Phys. Rev.* A **34** 1523
[11] Roux S, Hansen A, Lucena L, da Silva L and Pandey R 1991 *Preprint*